Brane Gravity from Bulk Vector Field

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Abstract

It is shown that there exists inverse to Kaluza-Klein possibility when Einstein's equations on the brane are received from Maxwell's multi-dimensional equations in flat space-time.

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Since the over 80 years old paper of Theodor Kaluza [1] it is assumed that geometrical gravity equations are more fundamental then matter fields equations. In standard Kaluza-Klein approach equations of matter fields are considered to be a part of multi-dimensional Einstein's equations. However, the problems of General Relativity are well known and even it is unclear is gravity quantum field or some classical effective interaction. Recently in the brane models it was shown that Plank's scale M_{Pl} and thus Newton's constant can be constructed with the fundamental scale and extra dimensional volume factor [2].

In the present paper we want to show that in the brane approach not only M_{Pl}, but Einstein equations as well possibly are effective. We claim that gravitation can be connected with the bulk vector field, which is the solution of Maxwell's equations. In this picture gravity exhibited tensor character only on the brane and graviton is constructed with two bulk spin 1 massless particles. This approach is inverse possibility of geometrisation when the vector field is fundamental and gravity is obtained from the brane geometry. In this direction geometrical unification of different interactions is easier, since Dirac equation also can be derived from the constrained Yang-Mills Lagrangian [3]. Similar ideas with inducing of gravity in the linear approximation on some plane in multi dimensions was considered in [4] using analogies with elasticity theory. We want to notice also the paper [5], which anticipates even General Relativity, where 4-dimensional gravity (but scalar) was assumed to be a remnant of 5-dimensional Maxwell's theory in flat space-time.

We want to begin with reminding that any n-dimensional Riemannian space can be embedded into N-dimensional pseudo-Euclidean space with $n \leq N \leq n(n+1)/2$ [6]. Thus, no more than ten dimensions are required to embed any 4-dimensional solution of Einstein's equations with arbitrary energy-momentum tensor. Embedding of the space-time with the coordinates x^α and metric $g_{\alpha\beta}$ into pseudo-Euclidean space with Cartesian coordinates ϕ^A and Minkowskian metric η_{AB} is given by

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = \eta_{AB}h^A_\alpha h^B_\beta dx^\alpha dx^\beta = \eta_{AB}d\phi^A d\phi^B \ . \tag{1}$$

Capital Latin letters $A, B, \ldots$ numerate coordinates of embedded space, while Greek indices $\alpha, \beta, \ldots$ numerate coordinates in four dimensions. Existence of the embedding (1) demonstrates that the tetrad field h^A_α can be expressed as a derivative of some vector

$$h^A_\alpha = \partial_\alpha \phi^A \ . \tag{2}$$

In four dimensions when tetrad index run over only four values such relation in general is impossible and according to (1) it could be always done in multi dimensions.

Suppose that in multi-dimensional flat space-time there exists (1+3)-brane with arbitrary geometry. We don't specify here the nature of the brane. For example, it can be the kink solution of nonlinear equation of some multi-dimensional scalar field. Not to care with extra indices and just to demonstrate the idea let us at first consider the case of bulk (1+4)-space with only one extra space-like dimension. Generalization for arbitrary dimensions and signature is obvious.

Let the equation of the branes surface in the Cartesian 5-dimensional coordinates X^A has the form

$$F(X^A) = 0 \ . \tag{3}$$

By introduction of the function

$$\xi(X^A) = F(X^A)/\sqrt{|\partial_B F \partial^B F|} \ , \tag{4}$$

metric of flat bulk space-time can be transformed to the Gaussian normal coordinates

$$ds^2 = -d\xi^2 + g_{\alpha\beta}(\xi, x^\nu) dx^\alpha dx^\beta \ . \tag{5}$$

Since $\xi = 0$ is the equation of the hyper-surface of the brane, the metric $g_{\alpha\beta}(0, x^\nu)$, which determines the geometry on the brane is the same 4-dimensional metric as used in (1) for the embedding.

By introducing of unit normal vector to the brane

$$n^A = \partial^A \xi \mid_{\xi=0} \ , \tag{6}$$

one can decompose by the standard way (see for example [7]) tensors of bulk space-time.

In the Gaussian system of coordinates (5) the Christoffel symbols on the brane are

$$\Gamma^\alpha_{\nu\lambda} = h^{A\alpha} \partial_\lambda h_{A\nu} \ . \tag{7}$$

Raising and lowering of Greek indices is made with $g_{\alpha\beta}$ and Latin indices with η_{AB}. The Christoffel symbols containing two or three indices ξ are equal to zero. The connections containing just one index ξ are forming outer curvature tensor of the brane, which using (2) can be written in the form

$$K_{\alpha\beta} = n_A D_\beta h^A_\alpha = -\Gamma^\xi_{\alpha\beta} = \partial_\alpha \partial_\beta \phi^\xi \ , \tag{8}$$

where D_β denotes covariant derivatives and ϕ^ξ is the transversal component of the embedding function.

Since bulk 5-dimensional space-time is pseudo-Euclidean its scalar curvature is zero

$${}^5R = R + K^2 - K_{\alpha\beta} K^{\alpha\beta} = 0 \ . \tag{9}$$

From this relation the 4-dimensional scalar curvature R can be expressed with quadratic combinations of the extrinsic curvature $K_{\alpha\beta}$. Thus, using (8) Hilbert's 4-dimensional gravitational action can be written in terms of the embedding variables

$$S_g = -M^2_{Pl} \int R\sqrt{-g} d^4x = M^2_{Pl} \int \left(\Box\phi^\xi \Box\phi^\xi - \partial_\alpha \partial_\beta \phi^\xi \partial^\alpha \partial^\beta \phi^\xi \right) \sqrt{-g} d^4x \ , \tag{10}$$

where $\Box = \partial_\alpha \partial^\alpha$ is the 4-dimensional wave operator. It is clear now that embedding theory allows us to rewrite 4-dimensional gravitational action in terms of brane derivatives of the normal components of some multi-dimensional vector.

Now let us consider the bulk massless vector field

$$A^B = \phi^B / \rho^{5/2} \ , \tag{11}$$

(ρ is some parameter with the dimension of X) which obeys 5-dimensional Maxwell's equations

$$\partial_A F^{AB} = 0 \ , \tag{12}$$

where $F_{AB} = \partial_A A_B - \partial_B A_A$. Here we don't want to identify the functions ϕ^A with the bulk coordinates X^A and restrict ourselves with pure geometrical interpretation.

Using integration by parts it is easy to show that Maxwell's action can be written in the form

$$S_\phi = -\frac{1}{4} \int F_{AB} F^{AB} d^5X = -\frac{1}{2\rho^5} \int [\partial_A \phi_B (\partial^A \phi^B + \partial^B \phi^A) - 2\partial_A \phi^A \partial_B \phi^B] d^5X \ , \tag{13}$$

We want to show that on the brane the action of vector field (13) can be reduced to the 4-dimensional gravity action (10).

In the Gaussian coordinates (5) Maxwell's equations (12) have the exact solution

$$\phi^\alpha = \xi\partial^\alpha\phi^\xi(x^\beta) \ , \qquad \phi^\xi = \phi^\xi(x^\beta) \ . \tag{14}$$

Inserting this solution into (13) and integrated by the normal coordinate ξ from $-\epsilon/2$ to $-\epsilon/2$, where ϵ is the brane width, we shall receive the induced action on the brane

$$S_\phi = -\frac{1}{\rho^5}\int \left[2\partial_\alpha\phi^\xi\partial^\alpha\phi^\xi + \xi^2(\partial_\alpha\partial_\beta\phi^\xi\partial^\alpha\partial^\beta\phi^\xi - \Box\phi^\xi\Box\phi^\xi)\right]\sqrt{g}d\xi d^4x =$$
$$= \frac{\epsilon^3}{12\rho^5}\int \left(\Box\phi^\xi\Box\phi^\xi - \partial_\alpha\partial_\beta\phi^\xi\partial^\alpha\partial^\beta\phi^\xi\right)\sqrt{-g}d^4x \ , \tag{15}$$

where g is determinant in Gaussian coordinates. If we put

$$M_{Pl}^2 = \epsilon^3/12\rho^5 \tag{16}$$

the effective action of 5-dimensional vector field (15) becomes equivalent to Hilbert action of 4-dimensional gravity (10). So, 4-dimensional Einstein's equations on the brane can be received from Maxwell's multi-dimensional equations in flat space-time.

The same result can be obtained in the general case $N > 5$. Now ξ and ϕ^ξ in (14) must be replaced with ξ^i and ϕ^i and action integral (15) transforms to the sum

$$S_{\Sigma\phi} = M^2\delta_{ij}\int \left(\Box\phi^i\Box\phi^j - \partial_\alpha\partial_\beta\phi^i\partial^\alpha\partial^\beta\phi^j\right)\sqrt{-g}d^4x \ , \tag{17}$$

where δij is Kronecker symbol and small Latin indices $i, j, \ldots$ numerates extra $(N-4)$ coordinates. If we assume that all the brane widths are equal to ϵ for the scale in (17) we have

$$M^2 = \epsilon^{(N-2)}/12\rho^N \ . \tag{18}$$

At the end we notice that the surface where Einstein's equations appear from the bulk vector field is not necessary to be 4-brane. For example, it can be 5-dimensional space with exponential warp factor needed to cancel cosmological constant [8]. Embedding of this metric was found in [9].

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